Vector Spaces Part 2

1. Linear Combination:

Given vectors $v_1, v_2, \ldots v_k$ in a vector space V and scalars $r_1, r_2, \ldots r_k$ in R, the vector $r_1v_1 + r_2v_2 + \ldots + r_kv_k$ is a linear combination of the vectors $v_1, v_2, \ldots v_k$ with scalar coefficients $r_1, r_2, \ldots r_k$.

2. Span:

Let X be a subset of a vector space V. The span of X, denoted by $sp(X)$, is the set of all linear combinations of vectors in X. If X is a finite set, like $\{v_1, v_2, \ldots v_k\}$, then we write $sp(X) = sp(v_1, v_2, \ldots v_k)$. If $W = sp(X)$, then we say that the vectors in X span or generate W. If $W = sp(X)$ and X is finite, then we say W is finitely generated.

3. Subset:

A subset W of a vector space V is closed under addition and scalar multiplication.

Let $u, v \in W$.
Let $r \in R$.

Closure under addition:
$u + v \in W$

Closure under multiplication:
$ru \in W$

4. Subspace:

A subspace is a subset with an extra requirement; it must be non-empty.

The conditions for a subspace are:

1. It must be non-empty.
2. Closed under addition.
3. Closed under scalar multiplication.

Note: A subspace of R^n always contains the origin.

Note: The subset $\{0\}$, which only contains the zero vector, is called the zero subspace.

Note: If V is a vector space and X is a non-empty subset of V, then $SP(X)$ is a subspace of V.

5. Independence:

Given a set of vectors, $\{x_1, x_2, \ldots x_k\}$, we can see if the vectors are linearly independent or linearly dependent.

If $r_1x_1 + r_2x_2 + \ldots r_kx_k = 0$ and $r_1 = r_2 = \ldots r_k = 0$, then $x_1, x_2, \ldots x_k$ are linearly independent.

Otherwise, they are linearly dependent.

E.g. 1 Show that $\{\sin x, \cos x\}$ is an independent set of functions.

Solution

$r(\sin x) + s(\cos x) = 0$

By subbing in various values for x, we can make a homogeneous system of equations. If that system only has the 0 solution, then the functions are independent. Otherwise, they are dependent.

$r(\sin(0)) + s(\cos(0)) = 0$

$r(\sin(\frac{\pi}{2})) + s(\cos(\frac{\pi}{2})) = 0$

$r(0) + s(1) = 0 \rightarrow s = 0$

$r(1) + s(0) = 0 \rightarrow r = 1$

$\therefore$ $\{\sin x, \cos x\}$ are lin indep.

E.g. 2 Show that $\{e^x, e^{2x}\}$ is an indep set of functions.

This time, we can differentiate e^x and e^{2x} to make a homogeneous system of equations.

$re^x + se^{2x} = 0$

$re^x + 2se^{2x} = 0$

Solving for this homogeneous system of equations, we get $r = s = 0$.

$\therefore$ e^x and e^{2x} are independent.

6. Basis:

Let V be a vector space. A set of vectors in V is a basis for V if:

1. The set of vectors span V.
2. The set of vectors is linearly independent.

7. Dimension:

Let V be a finitely generated vector space. The number of elements in a basis for V is the dimension of V, denoted as dim(V).

Note: Let V be a vector space. Let $w_1, w_2, \ldots w_k$ be vectors in V that span V, and let $v_1, v_2, \ldots v_m$ be vectors in V that are independent. Then, $k \geq m$.

Note: Let V be a finitely generated vector space. Then, any 2 bases of V have the same number of elements.